The Birthday Problem (Paradox)

What is the smallest group of random people needed so that it is likely (at least 50% probable) of a common birthday occurring within that group?

Tackling the problem

There are many ways a common birthday could occur.

Two or more people could share the same birthday, while everyone else has different birthdays.
Or the same as above, but more than one common birthday could be shared at the same time.

A simple way to solve this problem is to calculate the probability that all birthdays in the group are distinct. The complement of this event is that a common birthday has occurred.
Hence:

P(common birthday) = 1 – P(all distinct birthdays)

Calculating the probability of all distinct birthdays

Starting off with a group of 2 people, the chance of a common birthday will be: 364/365
This is because the 2^{nd} person has 364 possible days to have a birthday on out of 365.

Extending this to 3 people. Again we require the first two people have different birthdays.
We also require the 3^{rd} person to have a birthday different from the first 2.
So the probability of 3 distinct birthdays is: 364/365 * 363/365

For 4 people, this becomes: 364/365 * 363/365 * 362/365

We can now generalise this to $N \geq 2$ people:

364/365 * 363/365 * 362/365 * …. * (366-N)/365 ← There are N-1 factors here

Writing as a single fraction: $\frac{364 * 363 * 362 * \ldots * (366-N)}{365^{(N-1)}}$

Now multiply numerator and denominator by 365: $\frac{365 * 364 * 363 * 362 * \ldots * (366-N)}{365^N}$

Using factorial notation: eg 5! = 5 * 4 * 3 * 2 * 1 = 120

We have: $\frac{365!}{(365-N)! * 365^N}$

Let D = 365, to give $\frac{D!}{(D-N)! * D^N}$

Hence the probability of a common birthday is: $1 - \frac{D!}{(D-N)! * D^N}$